B.Sc. Semester-II Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 22112 **Course Code : SH/MTH/202/C-4**

Course Title : Group Theory-I

[NEW SYLLABUS]

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer any **five** from the following questions: 2×5=10

   a) Find all the elements of order 5 in the group $(\mathbb{Z}_{30},+)$.

   b) Give an example of a non-commutative group of order 18.

   c) Let G be a group and a mapping $\emptyset : G \to G$ be defined by $\emptyset(x) = x^{-1}$, $\forall x \in G$. Prove that $\emptyset$ is a homomorphism if and only if G is commutative.

   d) Show that the group $(\mathbb{Z},+)$ and $(\mathbb{Q},+)$ are not isomorphic.

[Turn Over]

   e) Show that number of elements of order 2 in a group of order 28 is odd.

   f) Let H and K are two subgroups of a finite group G such that $|H| > \sqrt{|G|}$ and $|K| > \sqrt{|G|}$. Prove that $|H \cap K| \geq 2$.

   g) Show that intersection of two normal subgroups is again normal.

   h) Let H be a subgroup of a commutative group G. Show that G/H is commutative.

UNIT-II

2. Answer any **four** from the following questions: 5×4=20

   a) $T_{ab}: \mathbb{R} \to \mathbb{R}$ defined by $T_{ab}(x) = ax + b$, $\forall x \in \mathbb{R}$ where $a, b \in \mathbb{R}$. Prove that

      i) $G = \{T_{ab}: a \neq 0\}$ forms a group under composition of mappings.

      ii) $H = \{T_{ab}: a = 1\}$ is a normal subgroups of G. 3+2

   b) Let $\emptyset : G \to G'$ be a homomorphism where $(G, o), (G', *)$ are two groups. Prove that $G/_{Ker\,\emptyset}$ is isomorphic to $\emptyset(G)$. 5

   c) Let G be a commutative group of order 17. Prove that the mapping $\emptyset : G \to G$ defined by $\emptyset(x) = x^8$, $x \in G$ is an isomorphism. 5

d) A cyclic group of finite order n has one and only one subgroup of order d for every positive divisor d of n. 5

e) i) Show that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

ii) Using first isomorphism theorem, show that $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*$, where $\mathbb{R}^*$ is the multiplicative group of all non-zero real numbers. 2+3

f) i) Prove that a non-commutative group of order 22 has a subgroup of order 11.

ii) Let G be a group of order 71. Show that the map $\phi: G \to G$, defined by $\phi(x) = x^{17}$ for all $x \in G$, is an isomorphism. 2+3

UNIT-III

3. Answer any **one** of the following questions:

10×1=10

a) i) If in a group $G, a^2 = e, \forall a \in G$ then prove that G is a commutative group, e being the identity element of G. Give an example of such group G with $o(G) > 4$.

ii) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group under component wise addition.

iii) Let H be a subgroup of a group G such that the product of any two left cosets of H is a left coset of H. Show that H is a normal subgroup of G. (2+2)+3+3

b) i) Let a be an element in a group G of order n. Show that $o(a^m) = \frac{o(a)}{\gcd(m,n)}$.

ii) Let G be a finite group of order 35. If G has two normal subgroups of orders 5 and 7, then show that G is cyclic.

iii) If a is the only element of order n in a group G, then show that $a \in Z(G)$.

iv) Let $G = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0\right\}$ and $H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$. Write down homomorphism (with justification) whose domain is G and kernel is H.

3+2+2+3

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